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Thermal lattice Bhatnagar-Gross-Krook model for flows with viscous heat dissipation in the incompressible limit

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In this paper, by introducing a different distribution function and starting from the Boltzmann equation as well as the Maxwell-Boltzmann distribution, we obtain a Boltzmann Bhatnagar-Gross-Krook (BGK) equation for thermal flows with viscous heat dissipation in the incompressible limit. The continuous thermal BGK model is then discretized over both time and phase space to form a lattice BGK model, which is shown to be consistent with some existing double distribution function lattice BGK models based on macroscopic governing equations. We have also demonstrated that the lattice BGK model derived theoretically in this work can be used to simulate laminar incompressible convention heat transfer with/without viscous heat dissipation.

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I. INTRODUCTION

The lattice Bhatnagar-Gross-Krook (LBGK) method has been widely used in various scientific and engineering computations over the last decade. As a mesoscopic numerical approach, the LBGK method numerically solves the kinetic equation (the Boltzmann BGK equation) [1,2] for the single-particle distribution function. In comparison with the conventional numerical algorithms, the kinetic features of the LBGK method enable it to be more effective for simulating complex fluid systems, such as flows in porous media [3], suspension flows [4], multiphase flows [5–12], and multicomponent flows [13–15].

Historically, the LBGK method, or more generally the lattice Boltzmann equation (LBE) method, originates from lattice gas automata (LGA), which mimic the microscopic dynamics of fluids by the motion of imaginary particles on regular lattices subject to some specific collision rules [16–18]. But later studies [1,2] showed that the LBGK method could be derived from the continuous Boltzmann BGK equation. Following this idea, various lattice BGK models for particular problems have been well constructed [6,24,27].

On the other hand, although the LBGK method has achieved great successes in simulating and modeling isothermal fluid dynamics problems, it is still a challenging problem to construct LBGK models for thermal flows with a solid physical foundation and with good numerical performance. Generally, the existing LBGK models for thermal flows in the literature fall into two categories: the multispeed models [19–24] and the double distribution function (DDF) models [25–29]. The multispeed models are an extension of the LBGK models for isothermal flows, in which only the single-particle distribution function f is defined and a higher order of velocity moment of this distribution function is used to describe the temperature field. In order to recover the macroscopic energy conservation equation, the multispeed LBGK models usually employ a larger set of discrete velocities than the corresponding isothermal models, and include higher order velocity terms in the equilibrium distribution function (EDF). Although the physical idea behind the multispeed LBGK models is straightforward and reasonable, it is rather onerous to derive the parameters in the EDF of such models. Another disadvantage of the multispeed LBGK models is that they usually suffer from severe numerical instability and is only suitable for problems with a rather narrow temperature range [21,22]. In addition, the multispeed LBGK models using a single relaxation time are limited to problems with a fixed Prandtl number, which departs far from the real physics. Although some methods have been proposed to overcome these problems (e.g., [22] for the improvement of the stability; [30] for the problem with variable Prandtl numbers), the drawbacks of the multispeed models still greatly limit their practical applications.

Alternatively, the double distribution function models utilize an additional distribution function, instead of the original single-particle distribution function, to describe the evolution of the temperature field [25–29]. It has been shown that the DDF LBGK models are simple and applicable to problems with different Prandtl numbers. More importantly, the DDF LBGK models have better numerical stability than the multispeed models. The reliability of the DDF models has been validated by many authors (e.g., [25–29]).

However, it should be pointed out most of the previous DDF LBGK models were proposed based on the observation that temperature is governed by an advection-diffusion equation under the conditions that both the compression work and viscous heat dissipation are negligible [25,26,28,29]. In these models, the temperature is regarded as a passive scalar. Early studies [26] even treated the temperature as a component of the mixture. As a result, the introduced temperature or internal energy distribution function became independent of the original distribution function. In the previous DDF LBGK models, the additional evolution equation and the corresponding EDF were developed rather heuristically as long as the macroscopic energy equation can be recovered. Furthermore, these models are usually constructed based on the as-

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sumption that both the compression work and the viscous heat dissipation can be neglected. Although it is reasonable to neglect the compression work in the incompressible limit, the viscous heat dissipation may have a significant effect on the temperature field, especially for flows of high Prandtl number fluids or flows with a high Eckert number. To our best knowledge, few of the previous DDF LBGK models, except the one proposed by He *et al.* [27], take into account the effect of viscous heat dissipation. However, the rest temperature EDF in the model by He *et al.* always takes negative values; and this model is rather complicated in comparison with other DDF models even in the case when the compression work and the viscous heat dissipation are negligible.

The objective of this work is to develop a lattice BGK model for thermal flows with viscous heat dissipation in the incompressible limit. To this end, we first introduce a distribution function related to the original single-particle distribution function to describe the temperature field, and then derive the continuous thermal lattice BGK model starting from the Boltzmann equation and the Maxwell-Boltzmann distribution, based on which a lattice BGK model is obtained for thermal flows with viscous heat dissipation in the incompressible limit. The proposed LBGK model is simple and robust in comparison with other existing DDF LBGK models for thermal flows with or without viscous heat dissipation in the incompressible limit. Moreover, we also show theoretically that most of the existing DDF LBGK models that were usually developed based on the macroscopic governing equations can be derived from our continuous model.

The rest of the article is organized as follows. In Sec. II, a temperature distribution function based on the original single-particle distribution function is defined, and its evolution equation together with the equilibrium distribution function is derived from the Boltzmann equation as well as Maxwell-Boltzmann distribution. In Sec. III, we derive the corresponding thermal lattice BGK model based on the results obtained in the previous sections. It is shown that many existing heuristic DDF LBGK models are consistent with our results. In Sec. IV, we employ our model to simulate several classical heat transfer problems, and finally some conclusions are drawn in Sec. V.

II. CONTINUOUS BGK EQUATIONS FOR THERMAL FLOWS

We start with the derivation of the continuous BGK equations for thermal flows from the continuous Boltzmann equation and the Maxwell-Boltzmann distribution.

A. Temperature distribution function and its evolution equation

In kinetic theory, the Boltzmann equation, which describes the evolution of the single-particle distribution function $\hat{f}(t,\vec{r},\vec{c})$, is given by [31]

$$\frac{\partial \hat{f}}{\partial t} + \vec{c}\cdot\frac{\partial \hat{f}}{\partial \vec{r}} = \frac{\partial_e \hat{f}}{\partial t}, \qquad (1)$$

where t, $\vec{r}$, and $\vec{c}$ denote the time, the particle position, and the particle velocity, respectively, and $\partial_e \hat{f}/\partial t$ is the rate of change in $\hat{f}$ due to binary collisions. Note that the effect of the external force field is not included in Eq. (1) (see [6,32] for the case when the effect of the external force field is included). The macroscopic variables, such as the density ρ, velocity $\vec{u}$, and temperature T, are defined, respectively, as

$$\rho = \int \hat{f}\, d\vec{c}, \qquad (2)$$

$$\rho\vec{u} = \int \hat{f}\vec{c}\, d\vec{c}, \qquad (3)$$

and

$$\frac{D\rho RT}{2} = \int \frac{1}{2}\hat{f}(\vec{c}-\vec{u})^2 d\vec{c}, \qquad (4)$$

where D is the space dimension. Here, without losing generality, we focus on three-dimensional problems $(D=3)$. Equations (2)–(4) indicate that the density ρ, velocity $\vec{u}$, and temperature T, are related, respectively, to the zeroth, the first, and the second moments of the distribution function $\hat{f}$ on the mesoscopic scale. Generally, the collision term $\partial_e \hat{f}/\partial t$ in the Boltzmann equation is rather complicated and hence simplification is needed in practical applications, provided that the basic features of $\partial_e \hat{f}/\partial t$ are retained. In fact, the well-known Bhatnagar-Gross-Krook model [33] originates from this need, which approximates the collision process as a relaxation to the local equilibrium

$$\frac{\partial_e \hat{f}}{\partial t} = -\frac{1}{\hat{\lambda}}(\hat{f} - \hat{f}^{eq}), \qquad (5)$$

where $\hat{\lambda}$ is the relaxation time and $\hat{f}^{eq}$ is the local Maxwell-Boltzmann equilibrium distribution function [31] given by

$$\hat{f}^{eq} = \rho(2\pi RT)^{-3/2}\exp[-(\vec{c}-\vec{u})^2/2RT], \qquad (6)$$

with R representing the gas constant.

It can be shown that the Boltzmann equation given by Eq. (1) with the BGK approximation given by Eq. (5) can recover the correct macroscopic continuity, momentum, and energy equations. However, the resulting Prandtl number of the system is fixed as a constant. This disadvantage is actually caused by the use of a single relaxation time to approximate the real collision process. In fact, as pointed out in [34], the relaxation time of energy carried by the particles to its equilibrium is different from that of momentum. This is why the original BGK model with a single relaxation time is inadequate to model a process involving both momentum and energy transport with different Prandtl numbers. This problem can be overcome by introducing two distinct relaxation times to characterize momentum and energy transport, respectively [34]. Motivated by this idea, we propose a two-relaxation-time, two-distribution-function BGK model which directly separates the energy transport from the momentum transport. In this model, we use a BGK equation with a relaxation time λ for a density distribution function f to model momentum transport, while using another BGK equation with a relaxation time λ_t for a different distribution function g to model energy transport. The first BGK equation describing momentum transport can be written as

$$\frac{\partial f}{\partial t} + \vec{c} \cdot \frac{\partial f}{\partial \vec{r}} = -\frac{1}{\lambda}(f - f^{eq}), \tag{7}$$

where

$$f^{eq} = \hat{f}^{eq} = \rho(2\pi RT)^{-3/2}\exp[-(\vec{c}-\vec{u})^2/2RT]. \tag{8}$$

Note that we have replaced $\hat{f}$ with f and $\hat{\lambda}$ with λ to distinguish the momentum BGK model given by Eqs. (7) and (8) from the original one, given by Eqs. (1) and (5), which describes both the momentum and energy transport with a single relaxation time.

To obtain the second BGK equation modeling energy transport, we first multiply both sides of the Boltzmann equation given by Eq. (1) by a factor, $(\vec{c}-\vec{u})^2/(3R)$, to give

$$\frac{\partial \hat{g}}{\partial t} + \vec{c} \cdot \frac{\partial \hat{g}}{\partial \vec{r}} = \frac{(\vec{c}-\vec{u})^2}{3R}\frac{\partial_e \hat{f}}{\partial t} + \hat{\mathfrak{R}}, \tag{9}$$

where

$$\hat{g} = \frac{(\vec{c}-\vec{u})^2}{3R}\hat{f}, \tag{10}$$

$$\hat{\mathfrak{R}} = -\hat{f}\frac{2(\vec{c}-\vec{u})}{3R} \cdot \left[\frac{\partial \vec{u}}{\partial t} + \left(\vec{c} \cdot \frac{\partial}{\partial \vec{r}}\right)\vec{u}\right]. \tag{11}$$

Equation (10) indicates that $\hat{g}$ is not an independent variable but related to the original single-particle distribution function $\hat{f}$. A comparison between Eqs. (4) and (10) shows that $\hat{g}$ represents energy carried by the particles. In this sense, Eq. (9) virtually describes the energy transport process. In a manner similar to the treatment of the collision operator for momentum transport, the collision integral in Eq. (9) can also be modeled by a BGK approximation with a relaxation time λ_t that represents the relaxation process of energy. With this approximation, we obtain another BGK equation:

$$\frac{\partial g}{\partial t} + \vec{c} \cdot \frac{\partial g}{\partial \vec{r}} = -\frac{1}{\lambda_t}(g - g^{eq}) + \mathfrak{R}, \tag{12}$$

where

$$g^{eq} = \frac{(\vec{c}-\vec{u})^2}{3R}\hat{f}^{eq} = \frac{(\vec{c}-\vec{u})^2\rho}{3R(2\pi RT)^{3/2}}\exp[-(\vec{c}-\vec{u})^2/2RT], \tag{13}$$

and $\mathfrak{R}$ has the same expression as $\hat{\mathfrak{R}}$ with $\hat{f}$ and $\hat{f}^{eq}$ replaced by f and f^{eq}, respectively. Note that the variable $\hat{g}$ in Eq. (9) is replaced by the variable g because of the BGK assumption in Eq. (12). The macroscopic variables can thus be redefined with f and g as

$$\rho = \int f\, d\vec{c}, \tag{14}$$

$$\rho\vec{u} = \int f\vec{c}\, d\vec{c}, \tag{15}$$

and

$$\rho T = \int g\, d\vec{c}. \tag{16}$$

Since Eq. (16) indicates that the temperature is the zeroth moment of g, hereafter we will refer to g as the temperature distribution function. It is worth mentioning that Eq. (4) can alternatively be rewritten as [27]

$$\rho\varepsilon = \int \frac{1}{2}\hat{f}(\vec{c}-\vec{u})^2 d\vec{c} = \int g'\, d\vec{c}, \tag{17}$$

where $g' = (\vec{c}-\vec{u})^2\hat{f}/2$. Equation (17) implies that the macroscopic internal energy ε includes the translational kinetic energy of the particles only. This is true for an ideal gas. For real gases, however, the internal energy must include the rotational kinetic energy, vibrational energy, and internal potential energy of the particles, in addition to the translational kinetic energy shown in Eq. (17). Therefore, Eq. (17) holds for an ideal gas only. However, the temperature distribution function g has no such limitation.

Equations (7), (8), (12), and (13) constitute a continuous Boltzmann BGK model for thermal flows. Through the Chapman-Enskog procedure [31], the macroscopic conservation equations of mass and momentum can be derived from Eq. (7):

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial \vec{r}} \cdot (\rho\vec{u}) = 0, \tag{18}$$

$$\frac{\partial(\rho\vec{u})}{\partial t} + \frac{\partial}{\partial \vec{r}} \cdot (\rho\vec{u}\vec{u}) = -\frac{\partial}{\partial \vec{r}}p + \frac{\partial}{\partial \vec{r}} \cdot \vec{\vec{\Pi}}, \tag{19}$$

and the macroscopic conservation equation of energy can be derived from Eq. (12):

$$\frac{\partial(\rho c_v T)}{\partial t} + \frac{\partial}{\partial \vec{r}} \cdot (\rho c_v \vec{u} T) = \frac{\partial}{\partial \vec{r}} \cdot \left(k\frac{\partial T}{\partial \vec{r}}\right) - p\left(\frac{\partial}{\partial \vec{r}} \cdot \vec{u}\right) + \vec{\vec{\Pi}} : \frac{\partial}{\partial \vec{r}}\vec{u}, \tag{20}$$

where the thermal conductivity $k = 5\lambda_t R^2 T/2$ and $\vec{\vec{\Pi}}$ represents the stress tensor.

The above analysis indicates that the thermal flows can be described by the two Boltzmann BGK-like equations Eqs. (7) and (12), which provide a base for developing thermal lattice Boltzmann BGK models. In fact, He *et al.* [27] has proposed such a model recently based on a similar BGK model. However, we notice that the BGK-like equation (12) can be further simplified. To this end, we rewrite Eq. (11) as $\mathfrak{R} = R^I + R^{II} + R^{III}$, with

$$R^I = -f\frac{2(\vec{c}-\vec{u})}{3R} \cdot \left[\frac{\partial \vec{u}}{\partial t} + \left(\vec{u} \cdot \frac{\partial}{\partial \vec{r}}\right)\vec{u}\right], \tag{21}$$

$$R^{II} = -f^{eq}\frac{2}{3R}(\vec{c}-\vec{u})(\vec{c}-\vec{u}):\frac{\partial}{\partial\vec{r}}\vec{u}, \tag{22}$$

and

$$R^{III} = -(f-f^{eq})\frac{2}{3R}(\vec{c}-\vec{u})(\vec{c}-\vec{u}):\frac{\partial}{\partial\vec{r}}\vec{u}. \tag{23}$$

It is well known that in the LBGK method the Mach number is required to be small, implying that the fluid is nearly incompressible. Under such a circumstance, the second term on the right hand side of Eq. (20), representing the compression work, can thus be neglected. Note that the compression work in Eq. (20) results virtually from the term R^{II} given by Eq. (22) [27]. Hence, for thermal flows at small Mach numbers considered in this work, Eq. (12) can be simplified as

$$\frac{\partial g}{\partial t} + \vec{c}\cdot\frac{\partial g}{\partial\vec{r}} = -\frac{1}{\lambda_t}(g-g^{eq}) + R^I + R^{III}. \tag{24}$$

Moreover, it can be shown that the zeroth moment of the term R^I vanishes, i.e.,

$$\int R^I d\vec{c} = 0,$$

which implies that R^I itself has no contribution to the temperature equation. However, it should be pointed out that the elimination of R^I would create an additional term, corresponding to the first moment of R^I, in the resulting macroscopic temperature equation. Further analysis indicates that this additional term is of order Ma^2 compared with the heat conduction term, which is negligible for low Mach number flows. Therefore, the final BGK equation for the temperature distribution function g now is simplified to

$$\frac{\partial g}{\partial t} + \vec{c}\cdot\frac{\partial g}{\partial\vec{r}} = -\frac{1}{\lambda_t}(g-g^{eq}) + R^{III}. \tag{25}$$

It is shown that in the incompressible limit we can use Eq. (25) to model the macroscopic energy equation for thermal flows with viscous heat dissipation that takes the same form as Eq. (20) but without the compression work term (see the Appendix for details).

It is noted that for low Prandtl number fluids or flows with small Eckert number, the viscous heat dissipation becomes less important and can also be neglected in many engineering applications. Under these situations, we can directly drop the term R^{III} in Eq. (25) to obtain

$$\frac{\partial g}{\partial t} + \vec{c}\cdot\frac{\partial g}{\partial\vec{r}} = -\frac{1}{\lambda_t}(g-g^{eq}), \tag{26}$$

which represents the evolution equation of the temperature distribution function for the case when the viscous heat dissipation is negligible.

B. The equilibrium distribution functions for low Mach number

In Sec. II A, we have developed a continuous thermal Boltzmann BGK model, i.e., Eq. (25), for thermal flows with viscous heat dissipation in the incompressible limit. We have also shown this model can further reduce to Eq. (26) for the case when the viscous heat dissipation is negligible. The EDF g^{eq} for the temperature distribution function g in both models is given by Eq. (13), which is now simplified as follows.

For low Mach number flows, the density EDF f^{eq} given by Eq. (8) and the temperature EDF g^{eq} given by Eq. (13) can be expanded as a Taylor series up to $\vec{u}^2$:

$$f^{eq} = \rho\left(\frac{1}{2\pi RT}\right)^{3/2}\exp\left(-\frac{\vec{c}^2}{2RT}\right)\left[1 + \frac{(\vec{c}\cdot\vec{u})}{RT} + \frac{(\vec{c}\cdot\vec{u})^2}{2R^2T^2} - \frac{\vec{u}^2}{2RT}\right] \tag{27}$$

and

$$g^{eq} = \rho T\left(\frac{1}{2\pi RT}\right)^{3/2}\exp\left(-\frac{\vec{c}^2}{2RT}\right)\left[\frac{\vec{c}^2}{3RT} + \left(\frac{\vec{c}^2}{3RT} - \frac{2}{3}\right)\frac{(\vec{c}\cdot\vec{u})}{RT} + \left(\frac{\vec{c}^2}{3RT} - \frac{4}{3}\right)\frac{(\vec{c}\cdot\vec{u})^2}{2R^2T^2} - \left(\frac{\vec{c}^2}{3RT} - \frac{2}{3}\right)\frac{\vec{u}^2}{2RT}\right]. \tag{28}$$

Since the polynomial of the density EDF f^{eq} given by Eq. (27) has been well documented, we focus on the discussion of the expanded series of g^{eq}. First, we regroup Eq. (28) as

$$\begin{aligned} g^{eq} = {} & \rho T\left(\frac{1}{2\pi RT}\right)^{3/2}\exp\left(-\frac{\vec{c}^2}{2RT}\right)\left[1 + \frac{(\vec{c}\cdot\vec{u})}{RT} + \frac{\vec{c}\vec{c}:\vec{u}\vec{u}}{2R^2T^2} - \frac{\vec{u}^2}{2RT}\right] + \rho T\left(\frac{1}{2\pi RT}\right)^{3/2}\exp\left(-\frac{\vec{c}^2}{2RT}\right)\left[\frac{\vec{c}^2}{3RT} - 1\right] \\ & + \rho T\left(\frac{1}{2\pi RT}\right)^{3/2}\exp\left(-\frac{\vec{c}^2}{2RT}\right)\left[\left(\frac{\vec{c}^2}{3RT} - \frac{5}{3}\right)\frac{(\vec{c}\cdot\vec{u})}{RT} + \left(\frac{\vec{c}^2}{3RT} - \frac{7}{3}\right)\frac{\vec{c}\vec{c}:\vec{u}\vec{u}}{2R^2T^2} - \left(\frac{\vec{c}^2}{3RT} - \frac{5}{3}\right)\frac{\vec{u}^2}{2RT}\right]. \end{aligned} \tag{29}$$

It can be readily proved that the zeroth through second order moments of the summed terms in the last square brackets on the right side of Eq. (29) vanish. Moreover, we can also prove that the zeroth and first order moments of the summed terms in the second square brackets in the right side of Eq. (29) are zero, and the second order moment is only related to the macroscopic thermal conductivity k. Thus, excluding this term only leads to a change in the thermal conductivity, from $k=5\rho\lambda_t R^2T/2$ to $k=\frac{3}{2}\rho\lambda_t R^2 T$. Clearly, the only difference is the constant in front of $\rho\lambda_t R^2 T$, which can be absorbed by manipulating the parameter, λ_t, in the numerical implementation. Therefore, we can drop the terms in the last two square brackets on the right side of Eq. (29) together to simplify Eq. (29) as

$$\begin{aligned} g^{eq} &= \rho T\left(\frac{1}{2\pi RT}\right)^{3/2}\exp\left(-\frac{\vec{c}^2}{2RT}\right) \times\left[1 + \frac{(\vec{c}\cdot\vec{u})}{RT} + \frac{(\vec{c}\cdot\vec{u})^2}{2R^2T^2} - \frac{\vec{u}^2}{2RT}\right] \\ &= Tf^{eq}, \end{aligned} \tag{30}$$

which indicates that the temperature EDF g^{eq} is related to the density EDF, f^{eq}, through the temperature T. On the other hand, it should be recognized that the inclusion of the terms in the second square brackets on the right side of Eq. (29) makes the temperature EDF negative for $\vec{c}=0$ [27], which contradicts the physical definition of a distribution function.

For low Mach number flows, the temperature EDF given Eq. (30) can further be simplified by neglecting the terms of $O(u^2)$ to give

$$g^{eq} = \rho T\left(\frac{1}{2\pi RT}\right)^{3/2}\exp\left(-\frac{\vec{c}^2}{2RT}\right)\left[1+\frac{(\vec{c}\cdot\vec{u})}{RT}\right]. \quad (31)$$

The simplification from Eq. (30) to Eq. (31) can be justified by comparing the zeroth to second order moments of the EDFs given by Eqs. (30) and (31). It can be proved that the zeroth and first order moments of both EDFs are identical; and the second order moments of the two EDFs differ only in terms with the order of Ma^2. Therefore, in the incompressible limit it is reasonable to neglect the terms of $O(u^2)$ in the EDF given by Eq. (30).

In summary, at this point we have obtained a continuous thermal BGK model for thermal flows in the incompressible limit, where the evolution equation for the density distribution function f is

$$\frac{\partial f}{\partial t} + \vec{c}\cdot\frac{\partial f}{\partial \vec{r}} = -\frac{1}{\lambda}(f-f^{eq}), \quad (32)$$

with

$$f^{eq} = \rho\left(\frac{1}{2\pi RT}\right)^{3/2}\exp\left(-\frac{\vec{c}^2}{2RT}\right) \times\left[1+\frac{(\vec{c}\cdot\vec{u})}{RT}+\frac{(\vec{c}\cdot\vec{u})^2}{2R^2T^2}-\frac{\vec{u}^2}{2RT}\right], \quad (33)$$

and the evolution equation for the temperature distribution function g is

$$\frac{\partial g}{\partial t} + \vec{c}\cdot\frac{\partial g}{\partial \vec{r}} = -\frac{1}{\lambda_t}(g-g^{eq}) + R^{III}, \quad (34)$$

with

$$R^{III} = -(f-f^{eq})\frac{2}{3R}(\vec{c}-\vec{u})(\vec{c}-\vec{u}):\frac{\partial}{\partial \vec{r}}\vec{u} \quad (35)$$

and

$$g^{eq} = \rho T\left(\frac{1}{2\pi RT}\right)^{3/2}\exp\left(-\frac{\vec{c}^2}{2RT}\right)\left[1+\frac{(\vec{c}\cdot\vec{u})}{RT}\right]. \quad (36)$$

For the case when the viscous heat dissipation is negligible, the evolution equation for the temperature distribution function g reduces to

$$\frac{\partial g}{\partial t} + \vec{c}\cdot\frac{\partial g}{\partial \vec{r}} = -\frac{1}{\lambda_t}(g-g^{eq}), \quad (37)$$

where the EDF g^{eq} is still given by Eq. (36).

III. THERMAL LATTICE BGK MODEL

In this section, we will discretize the Boltzmann BGK equations derived in the preceding section over the velocity space $V(\vec{c})$ and physical space $R(t,\vec{r})$ to form a lattice BGK model for thermal flows with viscous heat dissipation in the incompressible limit. First, we discuss the discretization of the velocity space. Following the procedure proposed by He and Luo [1,2], we first obtain the two-dimensional $(D=2)$ discrete velocity version of Eqs. (32) and (34) as

$$\frac{\partial f_i}{\partial t} + \vec{c}_i\cdot\frac{\partial f_i}{\partial \vec{r}} = -\frac{1}{\lambda}(f_i-f_i^{eq}) \quad (38)$$

and

$$\frac{\partial g_i}{\partial t} + \vec{c}_i\cdot\frac{\partial g_i}{\partial \vec{r}} = -\frac{1}{\lambda_t}(g_i-g_i^{eq}) + R_i^{III}, \quad (39)$$

where f_i and g_i are the distributions in terms of the discrete velocities. $R_i^{III}=-(1/R)(f_i-f_i^{eq})(\vec{c}_i-\vec{u})(\vec{c}_i-\vec{u}):\partial\vec{u}/\partial\vec{r}$ and the discrete velocities $\vec{c}_i$ are given by [35]

$$\vec{c}_i = \begin{cases} (0,0), & i=0, \\ c(\cos[(i-1)\pi/2],\sin[(i-1)\pi/2]), & i=1,2,3,4, \\ \sqrt{2}c(\cos[(2i-9)\pi/4],\sin[(2i-9)\pi/4]), & i=5,6,7,8, \end{cases} \quad (40)$$

with $c=\sqrt{3RT}$. The corresponding discrete EDFs f_i^{eq} and g_i^{eq} can be obtained from Eqs. (33) and (36) as follows:

$$f_i^{eq} = w_i\rho\left[1+\frac{(\vec{c}_i\cdot\vec{u})}{c_s^2}+\frac{(\vec{c}_i\cdot\vec{u})^2}{2c_s^4}-\frac{\vec{u}^2}{2c_s^2}\right] \quad (41)$$

and

$$g_i^{eq} = w_i\rho T\left[1+\frac{(\vec{c}_i\cdot\vec{u})}{c_s^2}\right], \quad (42)$$

where $c_s=\sqrt{RT}$ is the sound speed, and $w_0=4/9$, $w_i=1/9$ for $i=1$–4, and $w_i=1/36$ for $i=5$–8. Note that if we start from Eq. (30) instead of Eq. (36), the EDF g_i^{eq} will be obtained:

$$g_i^{eq} = w_i\rho T\left[1+\frac{(\vec{c}_i\cdot\vec{u})}{c_s^2}+\frac{(\vec{c}_i\cdot\vec{u})^2}{2c_s^4}-\frac{\vec{u}^2}{2c_s^2}\right], \quad (43)$$

which differs from that given by Eq. (42) in the last two terms of order Ma^2, just as in their continuous counterparts. The macroscopic variables are now defined by

$$\rho = \sum_{i=0}^{8} f_i, \quad \rho\vec{u} = \sum_{i=0}^{8} f_i\vec{c}_i, \quad \rho T = \sum_{i=0}^{8} g_i. \quad (44)$$

The above discretization procedures can also be extended to three-dimensional problems. Taking the three-dimensional 27-bit lattice model as an example, the corresponding discrete velocities are [1,2]

$$\vec{c}_i = \begin{cases} (0,0,0), & i=0, \\ (\pm 1,0,0)c,(0,\pm 1,0)c,(0,0,\pm 1)c, & i=1,\ldots,6 \\ (\pm 1,\pm 1,0)c,(\pm 1,0,\pm 1)c,(0,\pm 1,\pm 1)c, & i=7,\ldots,18 \\ (\pm 1,\pm 1,0)c, & i=19,\ldots,26 \end{cases} \quad (45)$$

and the weights appearing in the EDFs are $w_0 = 8/27$, $w_i = 2/27$ for $i=1-6$, $w_i = 1/54$ for $i=7-18$, and $w_i = 1/216$ for $i=19-26$. The term associated with the viscous heat dissipation in the discrete velocity BGK equation is given by

$$R_i^{III} = -(2/3R)(f_i - f_i^{eq})(\vec{c}_i - \vec{u})(\vec{c}_i - \vec{u}):\frac{\partial \vec{u}}{\partial \vec{r}}. \quad (46)$$

Note that the above discrete velocity evolution equations are still continuous in the physical space $R(t,\vec{r})$. Applying the first order forward Euler scheme in time and the first order upwind scheme in space to both equations (38) and (39), we obtain the following LBGK model:

$$f_i(t+\Delta t,\vec{r}+\vec{c}\Delta t) - f_i(t,\vec{r}) = -\omega[f_i(t,\vec{r}) - f_i^{eq}(t,\vec{r})], \quad (47)$$

$$g_i(t+\Delta t,\vec{r}+\vec{c}\Delta t) - g_i(t,\vec{r}) = -\omega_t[g_i(t,\vec{r}) - g_i^{eq}(t,\vec{r})] + \Delta t R_i^{III}, \quad (48)$$

where $\omega = \Delta t/\lambda$ is the dimensionless relaxation parameter for f_i and $\omega_t = \Delta t/\lambda_t$ is the dimensionless relaxation parameter for g_i.

Note that the above LBGK equations (47) and (48) are of only first order accuracy in both space and time for the continuous discrete velocity equations. However, the accuracy can be improved to second order by absorbing the first order discrete errors into the physical shear viscosity ν and the thermal conductivity k, respectively. In fact, we can derive the following macroscopic equations from the two LBGK equations (47) and (48) through the Chapman-Enskog procedure:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial \vec{r}} \cdot (\rho \vec{u}) = 0, \quad (49)$$

$$\frac{\partial (\rho \vec{u})}{\partial t} + \frac{\partial}{\partial \vec{r}} \cdot (\rho \vec{u}\vec{u}) = -\frac{\partial}{\partial \vec{r}} p + \frac{\partial}{\partial \vec{r}} \cdot (\rho \nu \vec{\vec{S}}), \quad (50)$$

$$\frac{\partial \rho c_\nu T}{\partial t} + \frac{\partial}{\partial \vec{r}} \cdot (\rho c_\nu \vec{u} T) = \frac{\partial}{\partial \vec{r}} \cdot \left(k \frac{\partial T}{\partial \vec{r}}\right) + (\rho \nu \vec{\vec{S}}):\frac{\partial}{\partial \vec{r}} \vec{u}, \quad (51)$$

where ν and k are now given by $(1/\omega - 0.5)c_s^2 \Delta t$ and $\rho c_\nu (1/\omega_t - 0.5)c_s^2 \Delta t$, respectively. Similarly, from Eqs. (32) and (37), we also obtain the following discrete velocity equations for thermal flows without viscous heat dissipation:

$$\frac{\partial f_i}{\partial t} + \vec{c}_i \cdot \frac{\partial f_i}{\partial \vec{r}} = -\frac{1}{\lambda}(f_i - f_i^{eq}) \quad (52)$$

and

$$\frac{\partial g_i}{\partial t} + \vec{c}_i \cdot \frac{\partial g_i}{\partial \vec{r}} = -\frac{1}{\lambda_t}(g_i - g_i^{eq}), \quad (53)$$

with the discrete EDFs f_i^{eq} and g_i^{eq} given by Eqs. (41) and (42), which can be discretized to obtain the following LBGK model:

$$f_i(t+\Delta t,\vec{r}+\vec{c}\Delta t) - f_i(t,\vec{r}) = -\omega[f_i(t,\vec{r}) - f_i^{eq}(t,\vec{r})], \quad (54)$$

$$g_i(t+\Delta t,\vec{r}+\vec{c}\Delta t) - g_i(t,\vec{r}) = -\omega_t[g_i(t,\vec{r}) - g_i^{eq}(t,\vec{r})]. \quad (55)$$

Note that in the incompressible limit, for thermal flows without viscous heat dissipation, the LBGK model given by Eqs. (54) and (55) together with the discrete EDFs f_i^{eq} and g_i^{eq} given by Eqs. (41) and (42) or (43) is consistent with those reported in the literature (e.g., [25,26,29]), which were constructed based on the macroscopic conservation equations, rather than starting from the Boltzmann BGK equation.

IV. NUMERICAL SIMULATIONS

In the previous section, we have derived a DDF LBGK model for incompressible thermal flows with viscous heat dissipation from the Boltzmann equation. In this section we shall apply the proposed model to simulate heat transfer in Couette flow and in Poiseuille flow with viscous heat dissipation to validate the accuracy of the model.

A. Heat transfer in Couette flow with the viscous heat dissipation

We first present the simulation results of heat transfer in Couette flow. Consider an incompressible and viscous fluid between two infinite parallel flat plates, separated by a distance of D. The upper plate at temperature T_h moves at speed U, and the lower plate at temperature T_l ($T_h > T_l$) is stationery. The exact solution of this problem is given by

$$\theta = y^* + \frac{\mathrm{Pr\,Ec}}{2} y^*(1-y^*), \quad (56)$$

where $\theta = (T-T_l)/(T_h-T_l)$, $y^* = y/D$; $\mathrm{Pr} = \nu/\alpha$ is the Prandtl number, and $\mathrm{Ec} = U^2/c_p(T_h - T_l)$ is the Eckert number, with α, ν, and c_p representing the thermal diffusivity, kinematic viscosity, and specific heat, respectively.

In simulations, periodic boundary conditions were applied to the inlet and outlet and the nonequilibrium extrapolation method [36] was applied to the two plates. The central difference scheme was adopted to discretize the term R_i^{III}.

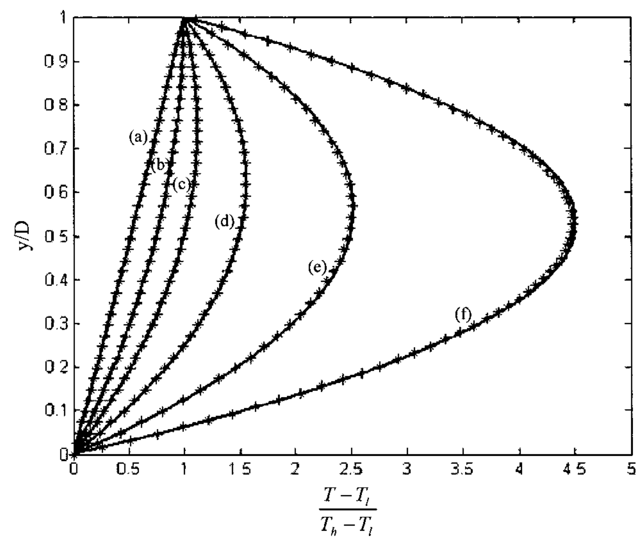

FIG. 1. Dimensionless temperature profile in Couette flow for Pr Ec=(a) 0.16, (b) 2.0, (c) 4.0, (d) 8.0, (e) 16.0, (f) 32. Solid line, analytical solution; *, numerical results.

We carried out the simulation on an 80×80 mesh for Ec=16.0 and Pr ranging from 0.01 to 1.0 and a 130×130 mesh for Ec=16.0 and Pr=2.0. The final dimensionless temperature profiles with different Pr Ec are compared with the exact solution, Eq. (56), in Fig. 1. It is shown that for small values of Pr Ec, temperature varies almost linearly along the direction perpendicular to the plates. This means that in comparison with heat conduction, the effect of the viscous heat dissipation is rather weak. As Pr Ec increases, the temperature profile deviates from the linear distribution. In particular, when Pr Ec$>$2.0, the maximum temperature of the fluid even exceeds the temperature at the upper plate. We can see from Fig. 1 that the numerical results by the model proposed in this work are in an excellent agreement with the exact solution. It is also worth mentioning that our model can also be used to simulate the problem at rather high Prandtl number. For instance, in Fig. 2 we show the numerical results for Pr=100 and Ec=1.0. It is seen that the numerical results are in good agreement with the analytical solution even for such a high Prandtl number fluid. This shows that our model possesses good numerical performance over a wide range of Prandtl number.

B. Heat transfer in Poiseuille flow with viscous heat dissipation

Poiseuille flow with viscous heat dissipation is another classical heat transfer problem. Unlike Couette flow, in this case, the parallel plates are all stationery and the incompressible and viscous fluid flow between the plates is driven by a constant pressure difference $-dp/dx$ along the direction parallel to the plates. It is well known that for Poiseuille flow with viscous heat dissipation, the macroscopic velocity distributes parabolically as

$$\frac{u}{U}=\frac{3}{2}\left[1-\left(\frac{2y}{D}-1\right)^2\right], \qquad (57)$$

and the temperature distributes as

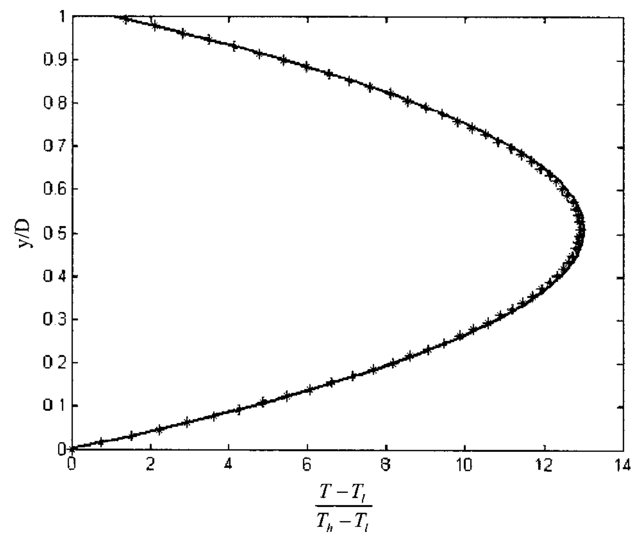

FIG. 2. Dimensionless temperature profile in Couette flow for Pr=100 and Ec=1.0. Solid line, analytical solution; *, numerical results.

$$\frac{T-T_l}{T_h-T_l}=\frac{y}{D}+\frac{3}{4}\mathrm{Pr\,Ec}\left[1-\left(\frac{2y}{D}-1\right)^4\right], \qquad (58)$$

where D represents the width between the plates, y represents the distance from the surface of the bottom plate, and $U=(dp/dx)D^2/12\mu$ is the average velocity. In simulations, we again applied the periodic boundary conditions to both the inlet and outlet of the channel, and the nonequilibrium extrapolation method to the two plates. As to f_i's at the surface of the plates, we adopted the bounce back rule. Meanwhile, the dimensionless relaxation time ω is chosen to be 1.25 and the density of the working fluid is set to be 1.0. Again, the central difference scheme was applied to discretize R_i^{III}.

We carried out simulations on an 80×80 mesh grid for Pr Ec=0.0, 0.375, 3.0, and a 130×130 mesh grid for Pr Ec =6.0. Figure 3 presents the comparison between the numerical results and the analytical solution, given by Eq. (58). From Fig. 3, it can be observed that the numerical results are in good agreement with the analytical solution for all the cases.

V. CONLUSION

In this paper, we have proposed a lattice BGK model for thermal flows with viscous heat dissipation in the incompressible limit. In this model, a temperature distribution function g is defined to represent the temperature field. The evolution equation and the EDF of this distribution function have been directly derived from the Boltzmann equation and the Maxwell-Boltzmann distribution. The resulting DDF LBGK model with two relaxation times can recover the correct macroscopic conservation equations for thermal flows with viscous heat dissipation in the incompressible limit and ensure that the EDF for each discrete velocity is nonnegative. We have further demonstrated that for the case when viscous heat dissipation is negligible, our thermal lat-

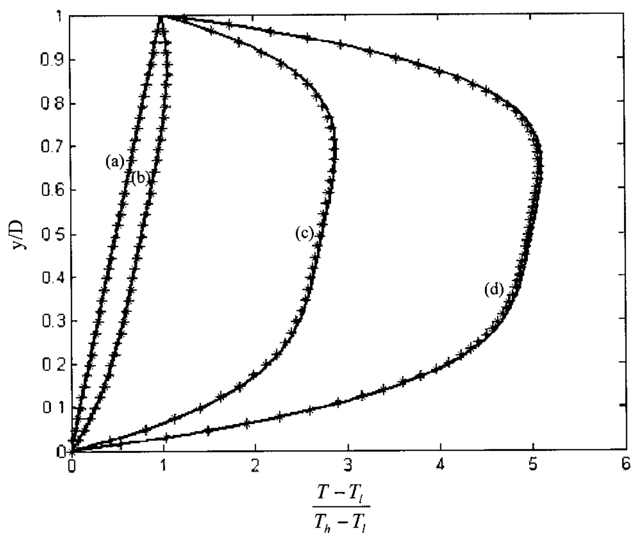

FIG. 3. Dimensionless temperature profile in Poiseuille flow for Pr Ec=(a) 0, (b) 0.375, (c) 3.0, (d) 6.0. Solid line, analytical solution; *, numerical results.

tice BGK model can reduce to a simple form, which is consistent with those reported in the literature. We have also carried out the numerical simulations of heat transfer in both Couette and Poiseuille flows to validate our thermal lattice BGK model. The numerical results are all in good agreement with the analytical solutions. It has also been shown that this thermal lattice BGK model can also be applied in rather high Prandtl number flows and gives a reasonable accuracy.

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APPENDIX: RECOVERING THE MACROSCOPIC CONSERVATION EQUATIONS THROUGH THE CHAPMAN-ENSKOG PROCEDURE

In the appendix, we present the detailed mathematic derivation of the macroscopic conservation equations of mass, momentum, and energy from Eqs. (7) and (25) through the Chapman-Enskog procedure [31]. Without loss of generality, we discuss the three-dimensional case. First, we introduce the following multiscale expansion:

$$f = f^{(0)} + \varepsilon f^{(1)} + \varepsilon^2 f^{(2)} + \cdots, \tag{A1}$$

$$g = g^{(0)} + \varepsilon g^{(1)} + \varepsilon^2 g^{(2)} + \cdots, \tag{A2}$$

$$\frac{\partial}{\partial t} = \varepsilon \frac{\partial_1}{\partial t} + \varepsilon^2 \frac{\partial_2}{\partial t}, \tag{A3}$$

$$\frac{\partial}{\partial \vec{r}} = \varepsilon \frac{\partial_1}{\partial \vec{r}}, \tag{A4}$$

where ε is a small parameter proportional to the Knudsen number. Substituting Eqs. (A1)–(A4) into Eqs. (7) and (25), we can obtain a series of equations in terms of the order of ε:

$$\varepsilon^0: \quad f^{(0)} = f^{eq}, \tag{A5}$$

$$\varepsilon^1: \quad \frac{\partial_1 f^{(0)}}{\partial t} + \vec{c} \cdot \frac{\partial_1 f^{(0)}}{\partial \vec{r}} = -\frac{1}{\lambda} f^{(1)}, \tag{A6}$$

$$\varepsilon^2: \quad \frac{\partial_2 f^{(0)}}{\partial t} + \frac{\partial_1 f^{(1)}}{\partial t} + \vec{c} \cdot \frac{\partial_1 f^{(1)}}{\partial \vec{r}} = -\frac{1}{\lambda} f^{(2)} \tag{A7}$$

and

$$\varepsilon^0: \quad g^{(0)} = g^{eq}, \tag{A8}$$

$$\varepsilon^1: \quad \frac{\partial_1 g^{(0)}}{\partial t} + \vec{c} \cdot \frac{\partial_1 g^{(0)}}{\partial \vec{r}} = -\frac{1}{\lambda_t} g^{(1)}, \tag{A9}$$

$$\varepsilon^2: \quad \frac{\partial_2 g^{(0)}}{\partial t} + \frac{\partial_1 g^{(1)}}{\partial t} + \vec{c} \cdot \frac{\partial_1 g^{(1)}}{\partial \vec{r}} = -\frac{1}{\lambda_t} g^{(2)} - f^{(1)} \frac{2}{3R} (\vec{c} - \vec{u})(\vec{c} - \vec{u}) : \frac{\partial_1}{\partial \vec{r}} \vec{u}. \tag{A10}$$

From Eqs. (A5) and (A8) and the definitions of f^{eq} and g^{eq} given by Eqs. (8) and (13), we can obtain the following moments of $f^{(r)}$ and $g^{(r)}$,

$$\int f^{(0)} d\vec{c} = \rho, \quad \int f^{(r)} d\vec{c} = 0 \text{ for } r = 1, 2, \ldots, \tag{A11}$$

$$\int f^{(0)} \vec{c}\, d\vec{c} = \rho \vec{u}, \quad \int f^{(r)} \vec{c}\, d\vec{c} = 0 \text{ for } r = 1, 2, \ldots, \tag{A12}$$

$$\int g^{(0)} d\vec{c} = \rho T, \quad \int g^{(r)} d\vec{c} = 0 \text{ for } r = 1, 2, \ldots, \tag{A13}$$

$$\int g^{(0)} \vec{c}\, d\vec{c} = \rho \vec{u} T, \tag{A14}$$

$$\int f^{(0)} \vec{c}\vec{c}\, d\vec{c} = \int f^{(0)} \vec{C}\vec{C}\, d\vec{c} + \rho \vec{u}\vec{u}, \quad \int f^{(0)} \vec{C}\vec{C}\, d\vec{c} = p\vec{\vec{I}}, \tag{A15}$$

where $\vec{C}$, the peculiar velocity, is defined as $\vec{c}$–$\vec{u}$.

With the help of Eqs. (A11)–(A14), we can obtain the macroscopic mass, momentum, and energy equations in the first order of ε by taking moments of Eq. (A6) and (A9) as

$$\frac{\partial_1 \rho}{\partial t} + \frac{\partial_1}{\partial \vec{r}} \cdot (\rho \vec{u}) = 0, \tag{A16}$$

$$\frac{\partial_1 (\rho \vec{u})}{\partial t} + \frac{\partial_1}{\partial \vec{r}} \cdot (\rho \vec{u}\vec{u}) = -\frac{\partial_1}{\partial \vec{r}} p, \tag{A17}$$

$$\frac{\partial_1(\rho c_\nu T)}{\partial t} + \frac{\partial_1}{\partial \vec{r}} \cdot (\rho c_\nu \vec{u} T) = 0, \tag{A18}$$

where $c_\nu = 3R/2$ is the specific heat at constant volume. On the other hand, from Eqs. (A6) and (A9) we can obtain

$$f^{(1)} = -\lambda\left(\frac{\partial_1 f^{(0)}}{\partial t} + \vec{c} \cdot \frac{\partial_1 f^{(0)}}{\partial \vec{r}}\right), \tag{A19}$$

$$g^{(1)} = -\lambda_t\left(\frac{\partial_1 g^{(0)}}{\partial t} + \vec{c} \cdot \frac{\partial_1 g^{(0)}}{\partial \vec{r}}\right). \tag{A20}$$

Using the results (A16)–(A18), and substituting Eqs. (8) and (13) into Eqs. (A19) and (A20), respectively, we can rewrite $f^{(1)}$ and $g^{(1)}$ as

$$f^{(1)} = -\lambda f^{(0)}\left\{\left[\frac{\vec{C}^2}{2RT} - \left(\frac{3}{2}+1\right)\right]\vec{C} \cdot \frac{\partial_1 \ln(T)}{\partial \vec{r}} + \left(\frac{\vec{C}\vec{C}}{RT} - \frac{\vec{C}^2\vec{\vec{I}}}{3RT}\right):\frac{\partial_1}{\partial \vec{r}}\vec{u}\right\} \tag{A21}$$

and

$$g^{(1)} = -\lambda_t g^{(0)}\left\{\left[\frac{\vec{C}^2}{2RT} - \left(\frac{3}{2}+1\right)\right]\vec{C} \cdot \frac{\partial_1 \ln(T)}{\partial \vec{r}} + \left(\frac{\vec{C}\vec{C}}{RT} - \frac{\vec{C}^2}{3RT}\vec{\vec{I}}\right):\frac{\partial_1}{\partial \vec{r}}\vec{u} - \frac{2\vec{C}}{\vec{C}^2}\cdot\left(\frac{\partial_1 \vec{u}}{\partial t} + \vec{u}\cdot\frac{\partial_1}{\partial \vec{r}}\vec{u}\right) - \frac{2\vec{C}\vec{C}}{\vec{C}^2}:\frac{\partial_1}{\partial \vec{r}}\vec{u}\right\}, \tag{A22}$$

where $\vec{\vec{I}}$ is the unity tensor.

With these results, we can obtain the macroscopic mass, momentum, and energy equations in the second order of ε by taking moments of Eq. (A7) and (A10) as

$$\frac{\partial_2 \rho}{\partial t} = 0, \tag{A23}$$

$$\frac{\partial_2(\rho \vec{u})}{\partial t} - \frac{\partial_1}{\partial \vec{r}} \cdot (\rho\nu)\vec{\vec{S}}_1 = 0, \tag{A24}$$

$$\frac{\partial_2(\rho c_\nu T)}{\partial t} = \frac{\partial_1}{\partial \vec{r}} \cdot \left(k\frac{\partial_1 T}{\partial \vec{r}} + F^I + \vec{u}F^{II}\right) + \vec{\vec{\Pi}}:\frac{\partial_1}{\partial \vec{r}}\vec{u} + \frac{\partial_1}{\partial t}F^{II}, \tag{A25}$$

where the second term on the right hand side of Eq. (A24), $-(\partial_1/\partial\vec{r})\cdot(\rho\nu)\vec{\vec{S}}_1$, results from $(\partial/\partial\vec{r})\cdot\int\vec{c}\vec{c}f^{(1)}d\vec{c}$, and denotes the stress experienced by the fluid; $\nu=\lambda RT$ is the shear viscosity, and $\vec{\vec{S}}_1 = \partial_1\vec{u}/\partial\vec{r} + (\partial_1\vec{u}/\partial\vec{r})^T - \frac{2}{3}[(\partial_1/\partial\vec{r})\cdot\vec{u}]\vec{\vec{I}}$ is the tensor associated with the velocity gradients. In Eq. (A25), $k = 5\rho\lambda_t R^2 T/2$ represents the thermal conductivity, and $\vec{\vec{\Pi}} = (\rho\nu)\vec{\vec{S}}_1$ represents the stress tensor. F^I and F^{II} are two additional terms caused by eliminating R^I and R^{II} from Eq. (12):

$$F^I = -\lambda_t \rho RT\left(\frac{\partial_1 \vec{u}}{\partial t} + \vec{u}\cdot\frac{\partial_1 \vec{u}}{\partial \vec{r}}\right), \tag{A26}$$

$$F^{II} = -\lambda_t p\frac{\partial_1}{\partial \vec{r}}\cdot\vec{u}. \tag{A27}$$

It is worth mentioning that the term of F^I is of order Ma^2 compared with the heat conduction. Hence, for thermal flows in the incompressible limit ($\mathrm{Ma} \ll 1$), this term can be dropped from Eq. (A25). As to F^{II}, the zeroth moment of $g^{(1)}$, it has been shown that it does not vanish but is related to the compression work only. In the incompressible limit, we can also neglect this term. Therefore, we can rewrite the macroscopic energy equation in the second order of ε as

$$\frac{\partial_2(\rho c_\nu T)}{\partial t} = \frac{\partial_1}{\partial \vec{r}}\cdot\left(k\frac{\partial_1 T}{\partial \vec{r}}\right) + \vec{\vec{\Pi}}:\frac{\partial_1}{\partial \vec{r}}\vec{u}. \tag{A28}$$

In summary, combining Eqs. (A16)–(A18), (A23), (A24), and (A28), we can obtain the macroscopic conservation equations of mass, momentum, and energy:

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial \vec{r}}\cdot(\rho\vec{u}) = 0, \tag{A29}$$

$$\frac{\partial(\rho\vec{u})}{\partial t} + \frac{\partial}{\partial \vec{r}}\cdot(\rho\vec{u}\vec{u}) = -\frac{\partial}{\partial \vec{r}}p + \frac{\partial}{\partial \vec{r}}\cdot\vec{\vec{\Pi}}, \tag{A30}$$

$$\frac{\partial(\rho c_\nu T)}{\partial t} + \frac{\partial}{\partial \vec{r}}\cdot(\rho c_\nu \vec{u}T) = \frac{\partial}{\partial \vec{r}}\cdot\left(k\frac{\partial T}{\partial \vec{r}}\right) + \vec{\vec{\Pi}}:\frac{\partial}{\partial \vec{r}}\vec{u}. \tag{A31}$$

[1] X. He and L. S. Luo, Phys. Rev. E **55**, R6333 (1997).
[2] X. He and L. S. Luo, Phys. Rev. E **56**, 6811 (1997).
[3] N. S. Martys and H. D. Chen, Phys. Rev. E **53**, 743 (1996).
[4] B. Chopard and A. Dupuis, Comput. Phys. Commun. **147**, 509 (2002).
[5] X. Shan and H. D. Chen, Phys. Rev. E **49**, 2941 (1994).
[6] X. Y. He, X. Shan, and G. D. Doolen, Phys. Rev. E **57**, R13 (1998).
[7] X. Y. He and G. D. Doolen, J. Stat. Phys. **107**, 309 (2002).
[8] L. S. Luo, Phys. Rev. Lett. **81**, 1618 (1998).
[9] L. S. Luo, Phys. Rev. E **62**, 4982 (2000).
[10] M. R. Swift, E. Orlandini, W. R. Osborn, and J. M. Yeomans, Phys. Rev. E **54**, 5041 (1996).
[11] R. Y. Zhang and H. D. Chen, Phys. Rev. E **67**, 066711 (2003).
[12] R. Y. Zhang, X. Y. He, and S. Y. Chen, Comput. Phys. Commun. **129**, 121 (2000).

[13] E. G. Flekkøy, Phys. Rev. E **47**, 4247 (1993).
[14] X. Shan and G. Doolen, J. Stat. Phys. **81**, 379 (1995).
[15] Z. L. Guo and T. S. Zhao, Phys. Rev. E **68**, 035302(R) (2003).
[16] U. Frisch, B. Hasslacher, and Y. Pomeau, Phys. Rev. Lett. **56**, 1505 (1986).
[17] R. Benzi, S. Succi, and M. Vergassola, Phys. Rep. **222**, 145 (1992).
[18] Sauro Succi, *The Lattice Boltzmann Equation for Fluid Dynamics and Beyond* (Clarendon, Oxford, 2001).
[19] F. J. Alexander, S. Chen, and J. D. Sterling, Phys. Rev. E **47**, R2249 (1993).
[20] Y. Chen, H. Ohashi, and M. Akiyama, Phys. Rev. E **50**, 2776 (1994).
[21] G. R. McNamara, A. L. Garcia, and B. J. Alder, J. Stat. Phys. **81**, 395 (1995).
[22] P. Pavlo, G. Vahala, and L. Vahala, Phys. Rev. Lett. **80**, 3960 (1998).
[23] G. Vahala, P. Pavlo, L. Vahala, and N. S. Martys, Int. J. Mod. Phys. C **9**, 1247 (1998).
[24] P. Pavlo, G. Vahala, and L. Vahala, J. Stat. Phys. **107**, 499 (2002).
[25] R. G. M. Van der Sman, Int. J. Mod. Phys. C **8**, 879 (1997).
[26] X. Shan, Phys. Rev. E **55**, 2780 (1997).
[27] X. Y. He, S. Y. Chen, and G. Doolen, J. Comput. Phys. **146**, 282 (1998).
[28] B. J. Palmer and D. R. Rector, J. Comput. Phys. **161**, 1 (2000).
[29] Z. L. Guo, B. C. Shi, and C. G. Zheng, Int. J. Numer. Methods Fluids **39**, 325 (2002).
[30] H. D. Chen, C. Teixeira, and K. Molvig, Int. J. Mod. Phys. C **8**, 675 (1997).
[31] S. Chapman and T. G. Cowling, *The Mathematical Theory of Non-Uniform Gases: An Account of the Kinetic Theory of Viscosity, Thermal Conduction and Diffusion in Gases* (Cambridge University Press, Cambridge, U.K., 1990).
[32] J. M. Buick and C. A. Greated, Phys. Rev. E **61**, 5307 (2000).
[33] P. L. Bhatnager, E. P. Gross, and M. Krook, Phys. Rev. **94**, 511 (1954).
[34] C. L. Woods, *An Introduction to the Kinetic Theory of Gases and Magnetoplasmas* (Oxford University Press, New York, 1993).
[35] Y. H. Qian, D. d'Humières, and P. Lallemand, Europhys. Lett. **17**, 479 (1992).
[36] Z. L. Guo, C. G. Zhang, and B. C. Shi, Phys. Fluids **14**, 2007 (2002).